Available online at www.sciencedirect.com

ScienceDirect

Computers and Mathematics with Applications 55 (2008) 2108–2117

An International Journal computers & mathematics with applications

www.elsevier.com/locate/camwa

Existence and uniqueness of periodic solutions for a kind of Rayleigh equation with two deviating arguments[☆]

Bingwen Liu

Department of Mathematics, Hunan University of Arts and Science, Changde, Hunan 415000, PR China

Received 24 September 2006; received in revised form 9 August 2007; accepted 30 August 2007

Abstract

In this paper, we consider a kind of Rayleigh equation with two deviating arguments of the form

$$x'' + f(t, x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = p(t).$$

By using the coincidence degree theory, we establish new results on the existence and uniqueness of periodic solutions for the above equation.

Keywords: Rayleigh equation; Deviating argument; Periodic solution; Coincidence degree

1. Introduction

Consider the Rayleigh equation with two deviating arguments of the form

$$x'' + f(t, x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = p(t), \tag{1.1}$$

where $\tau_i, p : R \to R$ and $f, g_i : R \times R \to R$ are continuous functions, τ_i and p are T-periodic, g_i is T-periodic in its first argument, $f(\cdot, 0) = 0$, $T > 0$ and $i = 1, 2$. In recent years, the problem of the existence of periodic solutions of Eq. (1.1) has been extensively studied in the literature. We refer the reader to [1–11] and the references cited therein. However, to the best of our knowledge, there exist no results for the existence and uniqueness of periodic solutions of Eq. (1.1).

The main purpose of this paper is to establish sufficient conditions for the existence and uniqueness of T-periodic solutions of Eq. (1.1). The results of this paper are new and complement previously known ones.

For ease of exposition, throughout this paper we will adopt the following notations:

$$|x|_k = \left(\int_0^T |x(t)|^k \mathrm{d}t \right)^{1/k}, \quad |x|_\infty = \max_{t \in [0,T]} |x(t)|.$$

☆ This work was supported by the NNSF (10371034) of China.
E-mail address: liubw007@yahoo.com.cn.

0898-1221/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2007.08.042

Let

$$X = \{x|x \in C^1(R, R), x(t+T) = x(t), \text{ for all } t \in R\}$$

and

$$Y = \{x|x \in C(R, R), x(t+T) = x(t), \text{for all } t \in R\}$$

be two Banach spaces with the norms

$$\|x\|_X = \max\{|x|_\infty, |x'|_\infty\}, \quad \text{and} \quad \|x\|_Y = |x|_\infty.$$

Define a linear operator $L : D(L) \subset X \longrightarrow Y$ by setting

$$D(L) = \{x|x \in X, x'' \in C(R,\ R)\}$$

and for $x \in D(L)$,

$$Lx = x''. \tag{1.2}$$

We also define a nonlinear operator $N : X \longrightarrow Y$ by setting

$$Nx = -f(t, x'(t)) - g_1(t, x(t - \tau_1(t))) - g_2(t, x(t - \tau_2(t))) + p(t). \tag{1.3}$$

It is easy to see that

$$\operatorname{Ker} L = R, \quad \text{and} \quad \operatorname{Im} L = \left\{x|x \in Y, \int_0^T x(s)\mathrm{d}s = 0\right\}.$$

Thus the operator L is a Fredholm operator with index zero.

Define the continuous projectors $P : X \longrightarrow \operatorname{Ker} L$ and $Q : Y \longrightarrow Y$ by setting

$$Px(t) = x(0) = x(T)$$

and

$$Qx(t) = \frac{1}{T}\int_0^T x(s)\mathrm{d}s.$$

Hence, $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Denoting by $L_P^{-1} : \operatorname{Im} L \longrightarrow D(L) \cap \operatorname{Ker} P$ the inverse of $L|_{D(L)\cap \operatorname{Ker} P}$, we have

$$L_P^{-1}y(t) = -\frac{t}{T}\int_0^T (t-s)y(s)\mathrm{d}s + \int_0^t (t-s)y(s)\mathrm{d}s. \tag{1.4}$$

It is convenient to introduce the following assumptions.

(A_0) assume that there exists a nonnegative constant C_1 such that

$$|f(t, x_1) - f(t, x_2)| \le C_1|x_1 - x_2|, \text{ for all } t, x_1, x_2 \in R.$$

$(\widetilde{A_0})$ assume that there exists a nonnegative constant C_2 such that

$$C_2|x_1 - x_2|^2 \le (x_1 - x_2)(f(t, x_1) - f(t, x_2)) \text{ for all } t, x_1, x_2 \in R.$$

2. Preliminary results

In view of (1.2) and (1.3), the operator equation $Lx = \lambda Nx$ is equivalent to the following equation

$$x'' + \lambda[f(t, x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t)))] = \lambda p(t), \tag{2.1_λ}$$

where $\lambda \in (0, 1)$.

For convenience of use, we introduce the Continuation Theorem [4, p. 40] as follows.

Lemma 2.1. *Let X and Y be two Banach spaces. Suppose that $L : D(L) \subset X \longrightarrow Y$ is a Fredholm operator with index zero and $N : X \longrightarrow Y$ is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. Moreover, assume that all the following conditions are satisfied:*

(1) *$Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$;*
(2) *$Nx \notin \operatorname{Im} L$, for all $x \in \partial\Omega \cap \operatorname{Ker} L$;*
(3) *The Brouwer degree*

$$\deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.$$

Then equation $Lx = Nx$ has at least one solution on $\overline{\Omega}$.

The following lemmas will be useful to prove our main results in Section 3.

Lemma 2.2. *If $x \in C^2(R, R)$ with $x(t + T) = x(t)$, then*

$$|x'(t)|_2^2 \leq \left(\frac{T}{2\pi}\right)^2 |x''(t)|_2^2. \tag{2.2}$$

Proof. Lemma 2.2 is a direct consequence of the Wirtinger inequality, and see [3] and [12] for its proof. □

Lemma 2.3. *Assume that the following conditions are satisfied.*

(A_1) *one of the following conditions holds:*
 (1) *$(g_i(t, u_1) - g_i(t, u_2))(u_1 - u_2) > 0$, for $i = 1, 2$, $u_i \in R$, $\forall t \in R$ and $u_1 \neq u_2$,*
 (2) *$(g_i(t, u_1) - g_i(t, u_2))(u_1 - u_2) < 0$, for $i = 1, 2$, $u_i \in R$, $\forall t \in R$ and $u_1 \neq u_2$;*

(A_2) *there exists a constant $d > 0$ such that one of the following conditions holds:*
 (1) *$x(g_1(t, x) + g_2(t, x) - p(t)) > 0$, for all $t \in R$, $|x| \geq d$,*
 (2) *$x(g_1(t, x) + g_2(t, x) - p(t)) < 0$, for all $t \in R$, $|x| \geq d$.*

If $x(t)$ is a T-periodic solution of $(2.1)_\lambda$, then

$$|x|_2 \leq \sqrt{T}d + \frac{T}{\pi}|x'|_2. \tag{2.3}$$

Proof. Let $x(t)$ be a T-periodic solution of $(2.1)_\lambda$. Set

$$x(t_{\max}) = \max_{t \in R} x(t), \qquad x(t_{\min}) = \min_{t \in R} x(t), \quad \text{where } t_{\max}, t_{\min} \in R.$$

Then we have

$$x'(t_{\max}) = 0, \quad x''(t_{\max}) \leq 0, \qquad \text{and} \qquad x'(t_{\min}) = 0, \quad x''(t_{\min}) \geq 0. \tag{2.4}$$

In view of $(2.1)_\lambda$, (2.4) implies that

$$g_1(t_{\max}, x(t_{\max} - \tau_1(t_{\max}))) + g_2(t_{\max}, x(t_{\max} - \tau_2(t_{\max}))) - p(t_{\max}) = -\frac{x''(t_{\max})}{\lambda} \geq 0, \tag{2.5}$$

and

$$g_1(t_{\min}, x(t_{\min} - \tau_1(t_{\min}))) + g_2(t_{\min}, x(t_{\min} - \tau_2(t_{\min}))) - p(t_{\min}) = -\frac{x''(t_{\min})}{\lambda} \leq 0. \tag{2.6}$$

Since $g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) - p(t)$ is a continuous function on R, it follows from (2.5) and (2.6) that there exists a constant $t_1 \in R$ such that

$$g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1) = 0. \tag{2.7}$$

Now we show that the following claim is true.

Claim. If $x(t)$ is a T-periodic solution of $(2.1)_\lambda$, then there exists a constant $t_2 \in R$ such that

$$|x(t_2)| \leq d. \tag{2.8}$$

Assume, by way of contradiction, that (2.8) does not hold. Then

$$|x(t)| > d, \text{ for all } t \in R, \tag{2.9}$$

which, together with (A_2) and (2.7), implies that one of the following relations holds:

$$x(t_1 - \tau_1(t_1)) > x(t_1 - \tau_2(t_1)) > d; \tag{2.10}$$

$$x(t_1 - \tau_2(t_1)) > x(t_1 - \tau_1(t_1)) > d; \tag{2.11}$$

$$x(t_1 - \tau_1(t_1)) < x(t_1 - \tau_2(t_1)) < -d; \tag{2.12}$$

$$x(t_1 - \tau_2(t_1)) < x(t_1 - \tau_1(t_1)) < -d. \tag{2.13}$$

Suppose that (2.10) holds, in view of $(A_1)(1)$, $(A_1)(2)$, $(A_2)(1)$ and $(A_2)(2)$, we will consider four cases as follows.
Case (i). If $(A_2)(1)$ and $(A_1)(1)$ hold, according to (2.10), we obtain

$$\begin{aligned} 0 &< g_1(t_1, x(t_1 - \tau_2(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1) \\ &< g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1), \end{aligned}$$

which contradicts (2.7). This contradiction implies that (2.8) is true.
Case (ii). If $(A_2)(1)$ and $(A_1)(2)$ hold, according to (2.10), we obtain

$$\begin{aligned} 0 &< g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_1(t_1))) - p(t_1) \\ &< g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1), \end{aligned}$$

which contradicts (2.7). This contradiction implies that (2.8) is true.
Case (iii). If $(A_2)(2)$ and $(A_1)(1)$ hold, according to (2.10), we obtain

$$\begin{aligned} 0 &> g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_1(t_1))) - p(t_1) \\ &> g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1), \end{aligned}$$

which contradicts (2.7). This contradiction implies that (2.8) is true.
Case (iv). If $(A_2)(2)$ and $(A_1)(2)$ hold, according to (2.10), we obtain

$$\begin{aligned} 0 &> g_1(t_1, x(t_1 - \tau_2(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1) \\ &> g_1(t_1, x(t_1 - \tau_1(t_1))) + g_2(t_1, x(t_1 - \tau_2(t_1))) - p(t_1), \end{aligned}$$

which contradicts (2.7). This contradiction implies that (2.8) is true.

Suppose that (2.11) (or (2.12), or (2.13)) holds, using methods similar to those used in Case (i)–Case (iv), we can show that (2.8) is true. This completes the proof of the above claim.

Let $t_2 = mT + t_0$, where $t_0 \in [0, T]$ and m is an integer. Then

$$|x(t_0)| \le d, \, |x|_\infty = \max_{t\in[0,T]} |x(t_0) + \int_{t_0}^{t} x'(s)\mathrm{d}s| \le d + \int_0^T |x'(s)|\mathrm{d}s \le d + \sqrt{T}|x'|_2. \tag{2.14}$$

Since $|x(t_0)| \le d$, by using the same approach used in the proof of Lemma 2.5 in [9], we get

$$|x|_2 \le \sqrt{T}d + \frac{T}{\pi}|x'|_2.$$

This completes the proof of Lemma 2.3. $\square$

Lemma 2.4 (*See [5]*). *Let $\mu \in [0,\ T]$ be a constant, $\overline{\delta} \in Y$, and $\sup_{t\in[0,\ T]} |\overline{\delta}(t)| \le \mu$. Then, for any $h \in X$,*

$$\int_0^T |h(s) - h(s - \overline{\delta}(s))|^2 \mathrm{d}s \le 2\mu^2 \int_0^T |h'(s)|^2 \mathrm{d}s. \tag{2.15}$$

Lemma 2.5. *Suppose that there exist a constant μ_i and an integer K_i such that*

(A*) $\mu_i = \sup_{t\in[0,T]} |\tau_i(t) - K_i T| \le T,\ i = 1, 2.$
Moreover, assume that (A_1) *and* (A_2) *hold, and one of the following conditions is satisfied:*

(A_3) *Suppose that* (A_0) *holds, and there exist nonnegative constants* b_1 *and* b_2 *such that*

$$\left[C_1 + b_1\left(\sqrt{2}\mu_1 + \frac{T}{\pi}\right) + b_2\left(\sqrt{2}\mu_2 + \frac{T}{\pi}\right)\right]\frac{T}{2\pi} < 1, \quad \text{and} \quad |g_i(t, x_1) - g_i(t, x_2)| \le b_i|x_1 - x_2|,$$

for all $t, x_1, x_2 \in R$, $i = 1, 2$;

(A_4) *Suppose that* $(\widetilde{\mathrm{A}}_0)$ *holds, and there exist nonnegative constants* b_1 *and* b_2 *such that*

$$0 \le b_1\left(\sqrt{2}\mu_1 + \frac{T}{\pi}\right) + b_2\left(\sqrt{2}\mu_2 + \frac{T}{\pi}\right) < C_2, \quad \text{and} \quad |g_i(t, x_1) - g_i(t, x_2)| \le b_i|x_1 - x_2|,$$

for all $t, x_1, x_2 \in R$, $i = 1, 2$. *Then Eq.* (1.1) *has at most one* T*-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of Eq. (1.1). Set $Z(t) = x_1(t) - x_2(t)$. Then, we obtain

$$\begin{aligned} &Z''(t) + (f(t, x_1'(t)) - f(t, x_2'(t))) + (g_1(t, x_1(t - \tau_1(t))) - g_1(t, x_2(t - \tau_1(t)))) \\ &\quad + (g_2(t, x_1(t - \tau_2(t))) - g_2(t, x_2(t - \tau_2(t)))) = 0. \end{aligned} \tag{2.16}$$

Set

$$Z(\bar{t}_1) = \max_{t\in R} Z(t), \qquad Z(\bar{t}_2) = \min_{t\in R} Z(t), \qquad \text{where } \bar{t}_1, \bar{t}_2 \in R.$$

Then, we have

$$Z'(\bar{t}_1) = x_1'(\bar{t}_1) - x_2'(\bar{t}_1) = 0, \quad Z''(\bar{t}_1) \le 0, \quad \text{and} \quad Z'(\bar{t}_2) = x_1'(\bar{t}_2) - x_2'(\bar{t}_2) = 0, \quad Z''(\bar{t}_2) \ge 0.$$

In view of (2.16), we obtain

$$g_1(\bar{t}_1, x_1(\bar{t}_1 - \tau_1(\bar{t}_1))) - g_1(\bar{t}_1, x_2(\bar{t}_1 - \tau_1(\bar{t}_1))) + g_2(\bar{t}_1, x_1(\bar{t}_1 - \tau_2(\bar{t}_1))) - g_2(\bar{t}_1, x_2(\bar{t}_1 - \tau_2(\bar{t}_1))) \ge 0$$

and

$$g_1(\bar{t}_2, x_1(\bar{t}_2 - \tau_1(\bar{t}_2))) - g_1(\bar{t}_2, x_2(\bar{t}_2 - \tau_1(\bar{t}_2))) + g_2(\bar{t}_2, x_1(\bar{t}_2 - \tau_2(\bar{t}_2))) - g_2(\bar{t}_2, x_2(\bar{t}_2 - \tau_2(\bar{t}_2))) \le 0.$$

Since

$$g_1(t, x_1(t - \tau_1(t))) - g_1(t, x_2(t - \tau_1(t))) + g_2(t, x_1(t - \tau_2(t))) - g_2(t, x_2(t - \tau_2(t)))$$

is a continuous function on R, it follows that there exists a constant $\bar{\xi} \in R$ such that

$$g_1(\bar{\xi}, x_1(\bar{\xi} - \tau_1(\bar{\xi}))) - g_1(\bar{\xi}, x_2(\bar{\xi} - \tau_1(\bar{\xi}))) + g_2(\bar{\xi}, x_1(\bar{\xi} - \tau_2(\bar{\xi}))) - g_2(\bar{\xi}, x_2(\bar{\xi} - \tau_2(\bar{\xi}))) = 0. \tag{2.17}$$

From (A_1), (2.17) implies that

$$Z(\bar{\xi} - \tau_1(\bar{\xi}))Z(\bar{\xi} - \tau_2(\bar{\xi})) = (x_1(\bar{\xi} - \tau_1(\bar{\xi})) - x_2(\bar{\xi} - \tau_1(\bar{\xi})))(x_1(\bar{\xi} - \tau_2(\bar{\xi})) - x_2(\bar{\xi} - \tau_2(\bar{\xi}))) \le 0.$$

Since $Z(t) = x_1(t) - x_2(t)$ is a continuous function on R, it follows that there exists a constant $\xi \in R$ such that

$$Z(\xi) = 0. \tag{2.18}$$

Let $\xi = nT + \widetilde{\gamma}$, where $\widetilde{\gamma} \in [0,\ T]$ and n is an integer. Then, (2.18) implies that there exists a constant $\widetilde{\gamma} \in [0,\ T]$ such that

$$Z(\widetilde{\gamma}) = Z(\xi) = 0. \tag{2.19}$$

Hence,

$$|Z(t)| = |Z(\widetilde{\gamma}) + \int_{\widetilde{\gamma}}^{t} Z'(s)\mathrm{d}s| \le \int_0^T |Z'(s)|\mathrm{d}s, \quad t \in [0,\ T],$$

and

$$|Z|_\infty \le \sqrt{T}|Z'|_2, \qquad |Z|_2 \le \frac{T}{\pi}|Z'|_2. \tag{2.20}$$

Now suppose that (A_3) (or (A_4)) holds, we shall consider two cases as follows.

Case (i). If (A_3) holds, multiplying $Z''(t)$ and (2.16) and then integrating it from 0 to T, together with (2.2), (2.15) and (2.20), we have

$$\begin{aligned}
|Z''|_2^2 &= \int_0^T |Z''(t)|^2 \mathrm{d}t \\
&= -\int_0^T (f(t, x_1'(t)) - f(t, x_2'(t)))Z''(t)\mathrm{d}t - \int_0^T (g_1(t, x_1(t-\tau_1(t))) - g_1(t, x_2(t-\tau_1(t))))Z''(t)\mathrm{d}t \\
&\quad - \int_0^T (g_2(t, x_1(t-\tau_2(t))) - g_2(t, x_2(t-\tau_2(t))))Z''(t)\mathrm{d}t \\
&\le C_1 \int_0^T |x_1'(t) - x_2'(t)|Z''(t)|\mathrm{d}t + b_1 \int_0^T |x_1(t-\tau_1(t)) - x_2(t-\tau_1(t))|Z''(t)|\mathrm{d}t \\
&\quad + b_2 \int_0^T |x_1(t-\tau_2(t)) - x_2(t-\tau_2(t))|Z''(t)|\mathrm{d}t \\
&\le C_1|Z'|_2|Z''|_2 + b_1 \int_0^T (|Z(t-\tau_1(t)+K_1T) - Z(t)| + |Z(t)|)|Z''(t)|\mathrm{d}t \\
&\quad + b_2 \int_0^T (|Z(t-\tau_2(t)+K_2T) - Z(t)| + |Z(t)|)|Z''(t)|\mathrm{d}t \\
&\le C_1|Z'|_2|Z''|_2 + b_1\sqrt{2}\mu_1|Z'|_2|Z''|_2 + b_1|Z|_2|Z''|_2 + b_2\sqrt{2}\mu_2|Z'|_2|Z''|_2 + b_2|Z|_2|Z''|_2 \\
&\le \left[C_1 + b_1\left(\sqrt{2}\mu_1 + \frac{T}{\pi}\right) + b_2\left(\sqrt{2}\mu_2 + \frac{T}{\pi}\right)\right]\frac{T}{2\pi}|Z''|_2^2.
\end{aligned} \tag{2.21}$$

Since $Z(t)$, $Z'(t)$ and $Z''(t)$ are T-periodic and continuous functions, in view of (A_3), (2.20) and (2.21), we have

$$Z(t) \equiv Z'(t) \equiv Z''(t) \equiv 0, \quad \text{for all } t \in R.$$

Thus, $x_1(t) \equiv x_2(t)$, for all $t \in R$. Therefore, Eq. (1.1) has at most one T-periodic solution.

Case (ii). If (A_4) holds, multiplying $Z'(t)$ and (2.16) and then integrating it from 0 to T, together with (2.15) and (2.20), we obtain

$$\begin{aligned}
C_2|Z'|_2^2 &= \int_0^T C_2|x_1'(t) - x_2'(t)|^2\mathrm{d}t \le \int_0^T f(t, x_1'(t) - f(t, x_2'(t)))(x_1'(t) - x_2'(t))\mathrm{d}t \\
&= -\int_0^T (g_1(t, x_1(t-\tau_1(t))) - g_1(t, x_2(t-\tau_1(t))))Z'(t)\mathrm{d}t \\
&\quad - \int_0^T (g_2(t, x_1(t-\tau_2(t))) - g_2(t, x_2(t-\tau_2(t))))Z'(t)\mathrm{d}t \\
&\le b_1 \int_0^T (|Z(t-\tau_1(t)+K_1T) - Z(t)| + |Z(t)|)|Z'(t)|\mathrm{d}t \\
&\quad + b_2 \int_0^T (|Z(t-\tau_2(t)+K_2T) - Z(t)| + |Z(t)|)|Z'(t)|\mathrm{d}t \\
&\le \left[b_1\left(\sqrt{2}\mu_1 + \frac{T}{\pi}\right) + b_2\left(\sqrt{2}\mu_2 + \frac{T}{\pi}\right)\right]|Z'|_2^2.
\end{aligned} \tag{2.22}$$

From (2.20) and (A_4), (2.22) implies that

$$Z(t) \equiv Z'(t) \equiv 0, \text{ for all } t \in R.$$

Hence, $x_1(t) \equiv x_2(t)$, for all $t \in R$. Therefore, Eq. (1.1) has at most one T-periodic solution. The proof of Lemma 2.5 is now complete. □

3. Main results

Theorem 3.1. *Let* (A^*), (A_1) *and* (A_2) *hold. Assume that either the condition* (A_3) *or the condition* (A_4) *is satisfied. Then Eq.* (1.1) *has a unique* T*-periodic solution.*

Proof. By Lemma 2.5, together with (A_3) and (A_4), it is easy to see that Eq. (1.1) has at most one T-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that Eq. (1.1) has at least one T-periodic solution. To do this, we shall apply Lemma 2.1. Firstly, we will claim that the set of all possible T-periodic solutions of $(2.1)_\lambda$ are bounded. In view of (A_3) and (A_4), we consider two cases as follows.

Case (1). If (A_3) holds. Let $x(t)$ be a T-periodic solution of $(2.1)_\lambda$. Multiplying $x''(t)$ and $(2.1)_\lambda$ and then integrating it from 0 to T, in view of (2.2), (2.3) and (2.15) and (A_3), we have

$$\begin{aligned}
|x''|_2^2 &= \int_0^T |x''(t)|^2 \mathrm{d}t \\
&= -\lambda \int_0^T f(t, x'(t))x''(t)\mathrm{d}t - \lambda \int_0^T g_1(t, x(t-\tau_1(t)))x''(t)\mathrm{d}t \\
&\quad -\lambda \int_0^T g_2(t, x(t-\tau_2(t)))x''(t)\mathrm{d}t + \lambda \int_0^T p(t)x''(t)\mathrm{d}t \\
&\le C_1 \int_0^T |x'(t)| x''(t)|\mathrm{d}t + \int_0^T [|g_1(t, x(t-\tau_1(t))) - g_1(t,0)| + |g_1(t,0)|] \cdot |x''(t)|\mathrm{d}t \\
&\quad + \int_0^T [|g_2(t, x(t-\tau_2(t))) - g_2(t,0)| + |g_2(t,0)|] \cdot |x''(t)|\mathrm{d}t + \int_0^T |p(t)| \cdot |x''(t)|\mathrm{d}t \\
&\le C_1 |x'|_2 |x''|_2 + b_1 \int_0^T |x(t-\tau_1(t))| \cdot |x''(t)|\mathrm{d}t + b_2 \int_0^T |x(t-\tau_2(t))| \cdot |x''(t)|\mathrm{d}t \\
&\quad + \int_0^T (|g_1(t,0)| + |g_2(t,0)|) \cdot |x''(t)|\mathrm{d}t + \int_0^T |p(t)| \cdot |x''(t)|\mathrm{d}t \\
&\le C_1 \frac{T}{2\pi} |x''|_2^2 + b_1 \int_0^T (|x(t-\tau_1(t)+K_1T) - x(t)| + |x(t)|) \cdot |x''(t)|\mathrm{d}t \\
&\quad + b_2 \int_0^T (|x(t-\tau_2(t)+K_2T) - x(t)| + |x(t)|) \cdot |x''(t)|\mathrm{d}t \\
&\quad + \int_0^T (|g_1(t,0)| + |g_2(t,0)| + |p(t)|) \cdot |x''(t)|\mathrm{d}t \\
&\le C_1 \frac{T}{2\pi} |x''|_2^2 + b_1 \sqrt{2}\mu_1 |x'|_2 |x''|_2 + b_1 |x|_2 |x''|_2 + b_2 \sqrt{2}\mu_1 |x'|_2 |x''|_2 + b_2 |x|_2 |x''|_2 \\
&\quad + [\max\{|g_1(t,0)| + |g_2(t,0)| : 0 \le t \le T\} + |p|_\infty] \sqrt{T} |x''|_2 \\
&\le \left[C_1 + b_1\left(\sqrt{2}\mu_1 + \frac{T}{\pi}\right) + b_2\left(\sqrt{2}\mu_2 + \frac{T}{\pi}\right)\right] \frac{T}{2\pi} |x''|_2^2 + [(b_1+b_2)d \\
&\quad + \max\{|g_1(t,0)| + |g_2(t,0)| : 0 \le t \le T\} + |p|_\infty] \sqrt{T} |x''|_2, \qquad (3.1)
\end{aligned}$$

which, together with (A_3), implies that there exist positive constants D_1 and D_2 such that

$$|x''|_2 < D_1, \qquad (3.2)$$

and

$$|x'|_2 < D_2, \qquad |x|_\infty < D_2. \qquad (3.3)$$

Since $x(0) = x(T)$, there exists a constant $\zeta \in [0, T]$ such that

$$x'(\zeta) = 0,$$

and

$$|x'(t)| = |x'(\zeta) + \int_{\zeta}^{t} x''(s)\mathrm{d}s| \le \sqrt{T}|x''|_2 < \sqrt{T}D_1 \quad \text{for all } t \in [0,\ T]. \tag{3.4}$$

Case (2). If (A_4) holds. Let $x(t)$ be a T-periodic solution of $(2.1)_\lambda$. Multiplying $x'(t)$ and $(2.1)_\lambda$ and then integrating it from 0 to T, by (A_4), (2.3) and (2.15) and the inequality of Schwarz, we have

$$\begin{aligned}
C_2|x'|_2^2 &= \int_0^T C_2 x'(t)x'(t)\mathrm{d}t \le \int_0^T f(t, x'(t))x'(t)\mathrm{d}t \\
&= -\int_0^T g_1(t, x(t-\tau_1(t)))x'(t)\mathrm{d}t - \int_0^T g_2(t, x(t-\tau_2(t)))x'(t)\mathrm{d}t + \int_0^T p(t)x'(t)\mathrm{d}t \\
&\le b_1 \int_0^T (|x(t-\tau_1(t)+K_1T) - x(t)| + |x(t)|)\cdot|x'(t)|\mathrm{d}t + b_2 \int_0^T (|x(t-\tau_2(t)+K_2T) \\
&\quad - x(t)| + |x(t)|)\cdot|x'(t)|\mathrm{d}t + \int_0^T (|g_1(t,0)| + |g_2(t,0)| + |p(t)|)\cdot|x'(t)|\mathrm{d}t \\
&\le \left[b_1\left(\sqrt{2}\mu_1 + \frac{T}{\pi}\right) + b_2\left(\sqrt{2}\mu_2 + \frac{T}{\pi}\right)\right]|x'|_2^2 + [(b_1+b_2)d \\
&\quad + \max\{|g_1(t,0)| + |g_2(t,0)| : 0 \le t \le T\} + |p|_\infty]\sqrt{T}|x'|_2.
\end{aligned} \tag{3.5}$$

In view of (2.14) and (3.5), there exists a constant $\overline{D_2} > 0$ such that

$$|x'|_2 < \overline{D_2}, \quad |x|_\infty = \max_{t\in[0,T]} |x(t_0) + \int_{t_0}^{t} x'(s)\mathrm{d}s| \le d + \sqrt{T}|x'|_2 < \overline{D_2}. \tag{3.6}$$

Multiplying $x''(t)$ and $(2.1)_\lambda$ and then integrating it from 0 to T, by (A_4), (2.3) and (3.1) and the inequality of Schwarz from (3.6), we obtain

$$\begin{aligned}
|x''|_2^2 &= C_1|x'|_2|x''|_2 + \int_0^T |x''(t)|^2\mathrm{d}t \\
&\le C_1\frac{T}{2\pi}|x''|_2^2 + b_1\int_0^T |x(t-\tau_1(t))|\cdot|x''(t)|\mathrm{d}t + b_2\int_0^T |x(t-\tau_2(t))|\cdot|x''(t)|\mathrm{d}t \\
&\quad + \int_0^T (|g_1(t,0)| + |g_2(t,0)|)\cdot|x''(t)|\mathrm{d}t + \int_0^T |p(t)|\cdot|x''(t)|\mathrm{d}t \\
&\le C_1\frac{T}{2\pi}|x''|_2^2 + (b_1+b_2)\sqrt{T}\overline{D_2}|x''|_2 + [\max\{|g_1(t,0)| + |g_2(t,0)| : 0 \le t \le T\} + |p|_\infty]\sqrt{T}|x''|_2,
\end{aligned}$$

it follows from (3.4) that there exists a positive constant $\overline{D_1}$

$$|x'(t)| \le \sqrt{T}|x''|_2 \le \overline{D_1} \quad \text{for all } t \in [0,\ T]. \tag{3.7}$$

Therefore, in view of (3.3), (3.4), (3.6) and (3.7), there exists a positive constant $M_1 > \max\{\sqrt{T}D_1 + D_2, \overline{D_1} + \overline{D_2}\}$ such that

$$\|x\|_X \le |x|_\infty + |x'|_\infty < M_1.$$

If $x \in \Omega_1 = \{x | x \in \operatorname{Ker} L \cap X,\ \text{ and } Nx \in \operatorname{Im} L\}$, then there exists a constant M_2 such that

$$x(t) \equiv M_2, \quad \text{and} \quad \int_0^T [g_1(t, M_2) + g_2(t, M_2) - p(t)]\mathrm{d}t = 0. \tag{3.8}$$

Thus,

$$|x(t)| \equiv |M_2| < d, \quad \text{for all } x(t) \in \Omega_1. \tag{3.9}$$

Let $M = M_1 + d + 1$. Set

$$\Omega = \{x | x \in X,\ |x|_\infty < M, |x'|_\infty < M\}.$$

It is easy to see from (1.3) and (1.4) that N is L-compact on $\overline{\Omega}$. We have from (3.8) and (3.9) and the fact $M > \max\{M_1,\ d\}$ that the conditions (1) and (2) in Lemma 2.1 hold.

Furthermore, define continuous functions $H_1(x, \mu)$ and $H_2(x, \mu)$ by setting

$$H_1(x, \mu) = -(1-\mu)x - \mu \cdot \frac{1}{T}\int_0^T [g_1(t, x) + g_2(t, x) - p(t)]\mathrm{d}t, \mu \in [0\,1],$$

$$H_2(x, \mu) = (1-\mu)x - \mu \cdot \frac{1}{T}\int_0^T [g_1(t, x) + g_2(t, x) - p(t)]\mathrm{d}t, \mu \in [0\,1].$$

If $(A_2)(1)$ holds, then

$$xH_1(x, \mu) \neq 0, \quad \text{for all } x \in \partial\Omega \cap \operatorname{Ker} L.$$

Hence, using the homotopy invariance theorem, we have

$$\begin{aligned}\deg\{QN,\ \Omega \cap \operatorname{Ker} L, 0\} &= \deg\left\{-\frac{1}{T}\int_0^T [g_1(t, x) + g_2(t, x) - p(t)]\mathrm{d}t,\ \Omega \cap \operatorname{Ker} L,\ 0\right\} \\ &= \deg\{-x,\ \Omega \cap \operatorname{Ker} L, 0\} \neq 0.\end{aligned}$$

If $(A_2)(2)$ holds, then

$$xH_2(x, \mu) \neq 0, \quad \text{for all } x \in \partial\Omega \cap \operatorname{Ker} L.$$

Hence, using the homotopy invariance theorem, we obtain

$$\begin{aligned}\deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\left\{-\frac{1}{T}\int_0^T [g_1(t, x) + g_2(t, x) - p(t)]\mathrm{d}t, \Omega \cap \operatorname{Ker} L, 0\right\} \\ &= \deg\{x, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.\end{aligned}$$

In view of all the discussions above, we conclude from Lemma 2.1 that Theorem 3.1 is proved. □

4. Example and remark

Example 4.1. Let $g(t, x) = \frac{1}{6\pi}x$, for all $t \in R$, $x > 0$, and $g(t, x) = \frac{1}{6\pi}\arctan x$, for all $t \in R$, $x \leq 0$. Then the Rayleigh equation

$$x''(t) + \frac{1}{16}x'(t) + \frac{1}{16}\sin x'(t) + g(t, x)t - \sin^2(t) + g(t, x)t - \cos^2 t = \frac{1}{40}\mathrm{e}^{\cos t} \tag{4.1}$$

has a unique 2π-periodic solution.

Proof. By (4.1), we have $b = \frac{1}{6\pi}$, $C_1 = \frac{1}{8}$, $\tau_1(t) = \sin^2 t$, $\tau_2(t) = \cos^2 t$ and $p(t) = \frac{1}{40}e^{\cos t}$. It is obvious that the assumptions (A_1) and (A_3) hold. Hence, by Theorem 3.1, Eq. (4.1) has a unique 2π-periodic solution. □

Remark 4.1. Eq. (4.1) is a very simple version of Rayleigh equation. Since $f(x) = \frac{1}{16}x + \frac{1}{16}\sin x$, $\tau_1(t) = \sin^2 t$ and $\tau_2(t) = \cos^2 t$, all the results in [1–11] and the references therein cannot be applicable to Eq. (4.1) to obtain the existence and uniqueness of 2π-periodic solutions. This implies that the results of this paper are essentially new.

Acknowledgement

The author would like to express his sincere appreciation to the reviewer for his/her helpful comments for improving the presentation and quality of this paper.

References

[1] T.A. Burton, Stability and Periodic Solution of Ordinary and Functional Differential Equations, Academic Press, Orland, FL, 1985.

[2] Wanmin Xiong, Qiyuan Zhou, Bing Xiao, Yixuan Wang, Fei Long, Periodic solutions for a kind of Liénard equation with two deviating arguments, Nonlinear Anal. Real World Appl. 8 (3) (2007) 787–796.
[3] J. Mawhin, Periodic solutions of some Vector retarded functional differential equations, J. Math. Anal. Appl. 45 (1974) 588–603.
[4] R.E. Gaines, J. Mawhin, Coincide degree and nonlinear differential equations, in: Lecture Notes in Math., vol. 568, Spring-Verlag, 1977.
[5] Shiping Lu, Weigao Ge, Periodic solutions for a kind of Liéneard equations with deviating arguments, J. Math. Anal. Appl. 249 (2004) 231–243.
[6] Shiping Lu, Weigao Ge, Some new results on the existence of periodic solutions to a kind of Rayleigh equation with a deviating argument, Nonlinear Anal. 56 (2004) 501–514.
[7] Huang Xiankai, Z.G. Xiang, On existence of 2π-periodic solutions for delay Duffing equation $x'' + g(t, x(t - \tau(t))) = p(t)$, Chinese Sci. Bull. 39 (1994) 201–203.
[8] Genqiang Wang, A priori bounds for periodic solutions of a delay Rayleigh equation, Appl. Math. Lett. 12 (1999) 41–44.
[9] Bingwen Liu, Lihong Huang, Periodic solutions for a class of forced Liénard-type equations, Acta Math. Appl. Sin. Engl. Ser. 21 (2005) 81–92.
[10] Chuangxia Huang, Yigang He, Lihong Huang, Wen Tan, New results on the periodic solutions for a kind of Rayleigh equation with two deviating arguments, Math. Comput. Modelling 46 (5–6) (2007) 604–611.
[11] Qiyuan Zhou, Fei Long, Existence and uniqueness of periodic solutions for a kind of Liénard equation with two deviating arguments, J. Comput. Appl. Math. 206 (2) (2007) 1127–1136.
[12] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, Cambridge University Press, London, 1964.